Integrable tops and non-commutative torus

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We consider the hydrodynamics of the ideal fluid on a 2-torus and its Moyal deformations. The both type of equations have the form of the Euler-Arnold tops. The Laplace operator plays the role of the inertia-tensor. It is known that 2-d hydrodynamics is non-integrable. After replacing of the Laplace operator by a distinguish pseudo-differential operator the deformed system becomes integrable. It is an infinite rank Hitchin system over an elliptic curve with transition functions from the group of the non-commutative torus. In the classical limit we obtain an integrable analog of the hydrodynamics on a torus with the inertia-tensor operator $\bar{\partial}^2$ instead of the conventional Laplace operator $\partial\bar{\partial}$.

1. Introduction

The Euler-Arnold tops (EAT) are Hamiltonian systems defined on the coadjoint orbits of groups [1]. Particular examples of such systems are the Euler top related to SO(3), its SO(N) generalization [2] and the hydrodynamic of the ideal incompressible fluid on a space M. The corresponding group of the latter system is SDiff(M). We consider here the case $\dim(M) = 2$ and then restrict M to be a torus T^2.

EAT are completely determined by their Hamiltonians, since the Poisson structure is fixed to be related to the Kirillov-Kostant form on the coadjoint orbits. The Hamiltonians are determined by the inertia-tensor operator $\mathfrak{J}$ mapping the Lie algebra $\mathfrak{g}$ to the Lie coalgebra $\mathfrak{g}^*$. Special choices of $\mathfrak{J}$ lead to completely integrable systems (see review [3]). In the case of the 2d hydrodynamics $\mathfrak{J}$ is the Laplace operator and it turns out that the theory is non-integrable [4]. The goal of this paper are integrable models related to SDiff. Some integrable models related to SDiff were considered in [5–8].

These type of models can be described as the classical limit of integrable models when the commutators in the Lax equations are replaced by the Poisson brackets. This approach was proposed in Ref.[9], where the dispersionless KP hierarchy was constructed and later developed in numerous publications (see review [10]).

Here we use the same strategy defining an integrable system on the non-commutative torus (NCT) and then taking the classical limit to SDiff(T^2). The starting point in our construction is the integrable GL($N,\mathbb{C}$) EAT introduced in Ref.[11,12]. Their inertia-tensor operators depend on the module τ, $Im\tau > 0$ of an elliptic curve. This curve is the basic spectral curve in the Hitchin description of the model. We consider here a special limit $N \to \infty$ of GL($N,\mathbb{C}$) that leads to the group G_θ of NCT, where θ is the Planck constant.[1] The group G_θ is defined as the set of invertible elements of the NCT algebra $\mathcal{A}_\theta$. It can be embedded in GL(∞) and in this way G_θ can be described as a special limit of GL($N,\mathbb{C}$). We define EAT related to G_θ depending on a parameter τ. Then, we construct the Lax operator with the spectral parameter on an elliptic curve with the same parameter τ.

In the classical limit $\theta \to 0$, $G_\theta \to$SDiff(T^2) and the inertia-tensor operator $\mathfrak{J}$ takes the form $\bar{\partial}^2$. The conservation laws survive in this limit while commutators in the Lax hierarchy become the Poisson brackets. It turns out that the classical limit is essentially the same as the rational

[1]For Manakov's top $\lim N \to \infty$ was considered in Ref.[13].

limit of the basic elliptic curve, such that the product of the Planck constant θ and the half periods of the basic curve are constant. In this way in the classical limit NCT becomes dual to the basic elliptic curve.

2. The Lie algebra of the non-commutative torus

Here we reproduce some basic results about NCT and related to it Lie algebra Sin_θ [14].

1. Non-commutative torus.

Quantum torus $\mathcal{A}_\theta$ is the unital algebra with two generators (U_1, U_2) that satisfy the relation

$$U_1U_2 = \mathbf{e}(\theta)U_2U_1,\ \mathbf{e}(\theta) = e^{2\pi i\theta},\ \theta \in [0,1)\,. \quad (2.1)$$

Elements of $\mathcal{A}_\theta$ are the double sums

$$\mathcal{A}_\theta = \{x = \sum_{m,n\in\mathbb{Z}} a_{mn}U_1^mU_2^n,\ a_{mn}\in\mathbb{C}\}\,,$$

where a_{mn} are rapidly decreasing on the lattice $\mathbb{Z}^2$

$$\sup_{m,n\in\mathbb{Z}}(1+m^2+n^2)^k|a_{mn}|^2 < \infty, \text{ for } k\in\mathbb{N}\,. \quad (2.2)$$

The trace functional $\mathrm{tr}(x)$ on $\mathcal{A}_\theta$ is defined as $\mathrm{tr}(x) = a_{00}$. It is positive definite and satisfies the evident identities

$$\mathrm{tr}(x^*x) = \sum_{m,n\in\mathbb{Z}} |a_{mn}|^2 \geq 0\,,$$

$$\mathrm{tr}(1) = 1, \quad \mathrm{tr}(xy) = \mathrm{tr}(yx\,,$$

where the star involution acts as $U_a^* = U_a^{-1},\ a_{mn}^* = \bar{a}_{mn}$. The relation with the commutative algebra of smooth functions on the two dimensional torus

$$T^2 = \{\mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}\} \sim \{0 < x \leq 1,\, 0 < y \leq 1\}. \quad (2.3)$$

comes from the identification

$$U_1 \to \mathbf{e}(x), \quad U_2 \to \mathbf{e}(y), \quad (2.4)$$

and the multiplication on T^2 becomes the Moyal multiplication:

$$(f*g)(x,y) := fg + \quad (2.5)$$
$$\sum_{n=1}^\infty \frac{(\pi\theta)^n}{n!}\varepsilon_{r_1s_1}\ldots\varepsilon_{r_ns_n}(\partial^n_{r_1\ldots r_n}f)(\partial^n_{s_1\ldots s_n}g).$$

Another representation with the operators, that acts on the space of functions on $\mathbb{R}/\mathbb{Z}$, has the form

$$U_1 \to \exp(i\varphi),\ U_2 \to \mathbf{e}(\theta\partial_\varphi)\,. \quad (2.6)$$

Finally, we can identify U_1, U_2 with matrices in $\mathrm{GL}(\infty)$

$$U_1 \to Q,\ U_2 \to \Lambda\,, \quad (2.7)$$

where Q and Λ are defined as

$$Q = \mathrm{diag}(\mathbf{e}(j\theta)),\ \Lambda = E_{j,j+1},\ j\in\mathbb{Z}\,.$$

The trace functional in the Moyal identification (2.4) is the integral

$$\mathrm{tr}f = \int_{T^2} f dxdy = f_{00}\,. \quad (2.8)$$

2. Sin-algebra

Define the following quadratic combinations of the generators

$$T_{mn} = \frac{i}{2\pi\theta}\mathbf{e}\left(\frac{mn}{2}\theta\right)U_1^mU_2^n\,. \quad (2.9)$$

Their commutator has the form of the sin-algebra

$$[T_{mn}, T_{m'n'}] = \quad (2.10)$$
$$= \frac{1}{\pi\theta}\sin\pi\theta(mn' - m'n)T_{m+m',n+n'}\,.$$

We denote Sin_θ the Lie algebra of the series

$$\psi = \sum_{mn}\psi_{mn}T_{mn} \quad (2.11)$$

with rapidly decreasing coefficients (2.2) and the commutator (2.10).

In the Moyal representation (2.5) the commutator has the form

$$[f(x,y), g(x,y)] = \{f,g\}^*\,,$$

where

$$\{f,g\}^* = \frac{1}{\theta}(f*g - g*f)\,. \quad (2.12)$$

The algebra Sin_θ has a central extension $\widehat{Sin}_\theta$ that comes from the one-cocycles of $\mathcal{A}_\theta$. The corresponding additional term in (2.10) has the form

$$(am+bn)\delta_{m,-m'}\delta_{n,-n'}\,. \quad (2.13)$$

Let θ be a rational number $\theta = p/N$, where $p, N \in \mathbb{N}$ with $\gcd(p, N) = 1$. Then Sin_θ has an ideal

$$I_N = \{T_{mn} - T_{m+Nl,n+Nk},\ k, l \in \mathbb{Z}\}, \tag{2.14}$$

and the factor $Sin_{p/N}/I_N$ for odd N is isomorphic to $\mathrm{GL}(N, \mathbb{C})$. Similarly, we can consider the ideal

$$\hat{I}_N = \{T_{mn} - T_{m+Nl,n},\ l \in \mathbb{Z}\}, \tag{2.15}$$

in $\widehat{Sin}_\theta$. The factor $\widehat{Sin}_\theta/\hat{I}_N$ is isomorphic to the central extended loop algebra $\hat{L}(\mathrm{gl}(N, \mathbb{C}))$.

3. 2d-hydrodynamics on $\mathcal{A}_\theta$

1. 2-d hydrodynamics

Let $\mathbf{v} = (V_x, V_y)$ be the velocity vector of the ideal incompressible fluid on a compact manifold M ($\dim(M) = 2$, $\mathrm{div}\mathbf{v} = 0$). [2] The Euler equation for 2d hydrodynamics takes the form [1]

$$\partial_t \mathrm{curl}\mathbf{v} = \mathrm{curl}[\mathbf{v}, \mathrm{curl}\mathbf{v}]\,, \tag{3.1}$$

where $\mathrm{curl}\mathbf{v} = \partial_x V_y - \partial_y V_x$ is the vorticity of the vector field $\mathbf{v}$.

Let $\psi(x, y)$ be the stream function. It is the Hamiltonian generating the vector field $\mathbf{v}$

$$i_{\mathbf{v}} dx \wedge dy = d\psi\,.$$

In other words $V_x = -\partial_y\psi$, $V_y = \partial_x\psi$. There is a isomorphisms of the Lie algebra of vector fields $\mathrm{div}\mathbf{v} = 0$ on M and the Poisson algebra of the stream functions $\mathfrak{g} = \{\psi\}$ on M defined up to constants

$$\{\psi_{\mathbf{v}_1}, \psi_{\mathbf{v}_2}\} = \psi_{[\mathbf{v}_1,\mathbf{v}_2]}$$

Let $\mathfrak{g}^*$ be the dual space of distributions on M. The vorticity $\mathcal{S} = \mathrm{curl}\mathbf{v}$ of the vector field $\mathbf{v}$

$$\mathcal{S} = -\Delta\psi$$

can be considered as an element from $\mathfrak{g}^*$. The Euler equation (3.1) for $\mathcal{S}$ takes the form

$$\partial_t\mathcal{S} = \{\mathcal{S}, \psi\},\ \text{ or }\ \partial_t\mathcal{S} = \{\mathcal{S}, \Delta^{-1}\mathcal{S}\}\,. \tag{3.2}$$

[2]For simplicity we assume that the measure on M is $dx \wedge dy$, though all expressions can be written in a covariant way.

We can look on (3.2) as the Euler-Arnold equation for the rigid top related to the Lie algebra $\mathfrak{g}$, where the Laplace operator is the map

$$\Delta\ : \mathfrak{g} \to \mathfrak{g}^*$$

that plays the role of the inertia-tensor. The phase space of the system is a coadjoint orbit of the group of the canonical transformations $\mathrm{SDiff}(M) = \exp\{\psi, \cdot\}$. The equation (3.2) takes the form

$$\partial_t\mathcal{S} = \mathrm{ad}^*_{\nabla H}\mathcal{S}\,, \tag{3.3}$$

where $\nabla H = \frac{\delta H}{\delta\mathcal{S}} = \psi$ is the variation of the Hamiltonian

$$H = \frac{1}{2}\int_M \mathcal{S}\Delta^{-1}\mathcal{S} = \int_M \psi\Delta\psi\,. \tag{3.4}$$

There is infinite set of Casimirs defining the coadjoint orbits

$$C_k = \int_M \mathcal{S}^k\,. \tag{3.5}$$

Certainly, these integrals cannot provide the integrability of the system [4].

Consider a particular case, when M is a two-dimensional torus (2.3). In terms of the Fourier modes s_{mn} of the vorticity

$$\mathcal{S} = \sum_{mn} s_{mn}\mathbf{e}(mx + ny)$$

the Hamiltonian (3.4) takes the form

$$H = -\frac{1}{8\pi^2}\sum_{mn}\frac{1}{m^2 + n^2}s_{mn}s_{-m,-n}\,, \tag{3.6}$$

and we come to the equation

$$\partial_t s_{mn} = -\frac{1}{8\pi^2}\sum_{j,k}\frac{jn - km}{j^2 + k^2}s_{jk}s_{m-j,n-k}\,. \tag{3.7}$$

2. 2d hydrodynamics on non-commutative torus.

We can consider the similar construction by replacing the Poisson brackets by the Moyal brackets (2.12) [15,16]. Introduce the vorticity $\mathcal{S}$ as an element of Sin^*_θ

$$\mathcal{S} = \sum_{mn} s_{mn}T_{mn}. \tag{3.8}$$

The equation (3.2) takes the form

$$\partial_t \mathcal{S} = \{\mathcal{S}, \Delta^{-1}\mathcal{S}\}^* \,,$$

or for the Fourier modes

$$\partial_t s_{mn} = \qquad (3.9)$$

$$-\frac{1}{8\pi^3\theta}\sum_{j,k}\frac{\sin \pi\theta(jn-km)}{j^2+k^2} s_{jk}s_{m-j,n-k} \;.$$

This system is the Euler-Arnold top on the group G_θ of invertible elements of $\mathcal{A}_\theta$ and the coadjoint orbits are defined by the same Casimirs (3.5) as for SDiff(T^2).

4. Elliptic rotator on $\mathcal{A}_\theta$

Let $\wp$ be the Weierstrass function depending on the modular parameter τ, Im$\tau > 0$.

$$\wp(u;\tau) = \frac{1}{u^2} + \qquad (4.1)$$

$$+ {\sum_{j,k}}' \left(\frac{1}{(j+k\tau+u)^2} - \frac{1}{(j+k\tau)^2}\right) \;.$$

We replace the inverse inertia-tensor Δ^{-1} of the hydrodynamics on the pseudodifferential operator $\mathfrak{J}^{-1} = J :\; \mathcal{S} \to \psi$ acting in a diagonal way on the Fourier coefficients (3.8):

$$J :\; s_{mn} \to \wp\left[\begin{array}{c} m \\ n \end{array}\right] s_{mn} = \psi_{mn}. \qquad (4.2)$$

Here

$$\wp\left[\begin{array}{c} m \\ n \end{array}\right] = \wp\left((m+n\tau)\theta;\tau\right) \;.$$

We consider the Euler-Arnold top with the inertia-tensor defined by J^{-1} (4.2). It has the Hamiltonian

$$H_\theta = \frac{\theta^2}{2}\int_{T^2} \mathcal{S}J(\mathcal{S}) = \frac{\theta^2}{2}\int_{T^2} \psi J^{-1}(\psi) = \qquad (4.3)$$

$$= \frac{\theta^2}{2}\sum_{mn} \wp\left[\begin{array}{c} m \\ n \end{array}\right] s_{mn}s_{-m,-n} \,,$$

where the integral is the trace functional (2.8). The equation of motion in the form of the Moyal brackets has the standard form

$$\partial_t \mathcal{S} = \theta^2\{\mathcal{S}, J(\mathcal{S})\}^* \,, \qquad (4.4)$$

or for the Fourier components

$$\partial_t s_{mn} = \frac{\theta}{\pi}\sum_{j,k} s_{jk}s_{m-j,n-k} \times \qquad (4.5)$$

$$\times \wp\left[\begin{array}{c} j \\ k \end{array}\right] \sin \pi\theta(jn-km) \,.$$

Consider the classical limit $\theta \to 0$ of this system when the Moyal brackets in (4.4) pass to the standard Poisson brackets. In this case we come to the top on the group SDiff(T^2). Since

$$\lim_{\theta\to 0} \theta^2 \wp\left[\begin{array}{c} m \\ n \end{array}\right] = \frac{1}{(m+n\tau)^2}$$

the Hamiltonian (4.3) takes the form

$$H = \frac{1}{2}\int_{T^2} \psi(\bar\partial)^2\psi \,, \qquad (4.6)$$

where $\bar\partial = \frac{1}{\rho}(\tau\partial_x - \partial_y)$, $\rho = \tau - \bar\tau$. The operator $\bar\partial^2$ plays the role of the inertia-tensor. It replaces the Laplace operator $\Delta \sim \partial\bar\partial$ for $\tau = i$ ($\partial = \frac{1}{\rho}(-\bar\tau\partial_x + \partial_y)$) of the standard hydrodynamics.

5. Integrability of elliptic rotator on $\mathcal{A}_\theta$

1. The Lax pair.

We will prove here that the Hamiltonian system of the elliptic rotator (4.4), (4.5) have an infinite set of involutive integrals of motion in addition to the Casimirs (3.5). It will follow from the Lax form

$$\partial_t L = [L, M] \qquad (5.1)$$

of the equations (4.4), (4.5). To define the Lax operator we introduce the basic spectral curve

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \,, \qquad (5.2)$$

with the already defined modular parameter τ. The Lax operator $L(z)$ is $(1,0)$-form on the spectral parameter $z \in E_\tau$ taking value in the coalgebra Sin^*_θ.

Introduce the following functions

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)} \,, \qquad (5.3)$$

where $\vartheta(u)$ is the odd theta-function on E_τ,

$$\vartheta(z|\tau) = \mathbf{e}\left(\frac{\tau}{8}\right)\sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i(n(n+1)\tau + 2nz)} \,, \quad (5.4)$$

and

$$\varphi\left[\begin{array}{c} m \\ n \end{array}\right](z) = \mathbf{e}(-n\theta z)\phi(-(m+n\tau)\theta, z)\,, \qquad (5.5)$$

$$f\left[\begin{array}{c} m \\ n \end{array}\right](z) = \qquad (5.6)$$

$$= \mathbf{e}(-n\theta z)\partial_u\phi(u,z)|_{u=-(m+n\tau)\theta}\ .$$

Then the Lax operator takes the form

$$L = \sum_{mn} s_{mn}\varphi\left[\begin{array}{c} m \\ n \end{array}\right](z)T_{mn}\,, \qquad (5.7)$$

and

$$M = \sum_{mn} s_{mn} f\left[\begin{array}{c} m \\ n \end{array}\right](z)T_{mn}\,. \qquad (5.8)$$

The equivalence of (5.1) and (4.5) follows from the Calogero functional equation

$$\varphi\left[\begin{array}{c} m \\ j \end{array}\right] f\left[\begin{array}{c} j \\ n \end{array}\right] - \varphi\left[\begin{array}{c} j \\ n \end{array}\right] f\left[\begin{array}{c} m \\ j \end{array}\right] =$$

$$= \left(\wp\left[\begin{array}{c} m \\ j \end{array}\right] - \wp\left[\begin{array}{c} j \\ n \end{array}\right]\right)\varphi\left[\begin{array}{c} m \\ n \end{array}\right]\,,$$

and the identity

$$\varphi\left[\begin{array}{c} m \\ n \end{array}\right](z)\varphi\left[\begin{array}{c} -m \\ -n \end{array}\right](z) = \wp(z) - \wp\left[\begin{array}{c} m \\ n \end{array}\right]\,.$$

2. Hitchin description.

It turns out that the elliptic rotator is the Hitchin system [17,18] on E_τ. It was proved in [12] for the group $\mathrm{GL}(N,\mathbb{C})$. Here we replace $\mathrm{GL}(N,\mathbb{C})$ by the infinite-dimensional group G_θ. It means that we consider the infinite rank vector bundle P over E_τ with the structure group G_θ. The Hitchin systems of infinite rank were introduced in [12,19]. Here we reproduce the properties of the concrete Lax equation as the Hitchin system. The Lax operator plays the role of the Higgs field. It satisfies the Hitchin equation

$$\bar\partial L = 0, \quad Res\, L|_{z=0} = \mathcal{S},$$

with the quasi-periodicity conditions

$$L(z+1) = QL(z)Q^{-1},$$

$$L(z+\tau) = \Lambda L(z)\Lambda^{-1}\,.$$

It is easy to see that (5.7) does satisfy these conditions.

3. Integrals of motion.

In the Hitchin construction the integrals are defined by means of the $(-j,1)$-differentials $\mu_j \in \Omega^{(-j,1)}(E_\tau)$. Let us choose the representatives from $\Omega^{(-j,1)}(E_\tau)$ that form a basis in the cohomology space $H^1(E_\tau,\Gamma^j)$, $(\dim H^1 = j)$

$$\mu_j = (\mu_{0,j}\partial_z^{j-1}\otimes d\bar z, \mu_{2,j}\partial_z^{j-1}\otimes d\bar z \ldots \mu_{j,j}\partial_z^{j}\otimes d\bar z)\,.$$

Let $\chi(z,\bar z)$ be a characteristic function of a small neighborhood of $z=0$. We can choose $\mu_{s,j}$ in the form

$$\mu_{0,j} \sim \theta^j\bar\partial(\bar z - z)(1-\chi(z,\bar z))\,,^3$$

$$\mu_{s,j} = c_{s,j}z^{s-1}\bar\partial\chi(z,\bar z) \text{ for } j>1,\ j\geq s>1, \qquad (5.9)$$

$$c_{s,j} \sim \theta^{j-s}\,.$$

The integrals

$$I_{s,j} = \frac{1}{j}\int_{E_\tau}\int_{T^2}(L^j)\mu_{s,j} \qquad (5.10)$$

are well defined and generate the infinite set of conservation laws. Here we integrate over the basic spectral curve E_τ (5.2) and NCT $\mathcal{A}_\theta$. The conservation laws can be extracted from the expansion of the elliptic function

$$\frac{1}{j}\int_{T^2} L^j(z) = I_{0,j} + \sum_{r=2}^{j} I_{r,j}\wp^{(r-2)}(z),\ (j=2,\ldots)\,.$$

In particular,

$$\int_{T^2} L^2(z) = I_{0,2} + \wp(z)\int_{T^2}\mathcal{S}^2,\ H = I_{0,2}\,.$$

Note that

$$I_{j,j} \sim C_j = \int_{T^2}\mathcal{S}^j$$

are the Casimirs (3.5).

Consider, for example, the integrals, that have the third order in the field $\mathcal{S}$. Let

$$E_1(u) = \partial_u \log\vartheta(u)\,,$$

[3] $\mu_{0,2}$ is the Beltrami differential

be the first Eisenstein function.[4] The second Eisenstein function is

$$E_2(u) = \wp(u) + 2\zeta(1/2)\,.$$

For the functions $\phi(u,z)$ (5.3) we have the following relation. If $u_1 + u_2 + u_3 = 0$ then

$$\phi(u_1,z)\phi(u_2,z)\phi(u_3,z) = -\frac{1}{2}E_2'(z) + \tag{5.11}$$
$$E_2(z)\left(E_1(u_1)+E_1(u_2)+E_1(u_3)\right)+$$
$$E_2(u_3)\left(E_1(u_1)+E_1(u_2)+E_1(u_3)\right) - \frac{1}{2}E_2'(u_3)\,.$$

This relation allows to calculate $\int_{T^2} L^3(z)$. Let

$$E_2\left[\begin{array}{c} m \\ n \end{array}\right] = E_2((m+n\tau)\theta),$$

$$E_1\left[\begin{array}{c} m \\ n \end{array}\right] = E_1((m+n\tau)\theta)\,.$$

Then in terms of the Fourier modes $\mathcal{S} = \{s_{mn}\}$ the integrals take the form

$$I_{1,3} = \frac{\theta^3}{3}\sum_{\sum m_j = \sum n_j = 0}\prod_{j=1}^{3} s_{m_j n_j}\times \tag{5.12}$$
$$\left(E_2\left[\begin{array}{c} m_3 \\ n_3 \end{array}\right]\sum_j E_1\left[\begin{array}{c} m_j \\ n_j \end{array}\right] - \frac{1}{2}E_2'\left[\begin{array}{c} m_3 \\ n_3 \end{array}\right]\right),$$

$$I_{2,3} = -\frac{\theta}{6}\sum_{\sum m_j = \sum n_j = 0}\prod_{j=1}^{3} s_{m_j n_j}\times \tag{5.13}$$
$$\times\sum_j E_1\left[\begin{array}{c} m_j \\ n_j \end{array}\right]\,.$$

The integrals (5.10) allow to write the hierarchy of the commuting flows

$$\partial_{s,j}\mathcal{S} = \{\nabla I_{s,j}, \mathcal{S}\}^* \quad (\partial_{s,j} = \partial_{t_{s,j}})\,. \tag{5.14}$$

They have the Lax representation with L (5.7)

$$\partial_{s,jL} = [L, M_{s,j}] := \{L, M_{s,j}\}^*\,, \tag{5.15}$$

and $M_{s,j}$ is partly fixed by the equation

$$\bar{\partial} M_{s,j} = -L(z)^{j-1}\mu_{s,j}\,.$$

[4] It is related to the Weierstrass zeta-function as $E_1(u) = \zeta(u) - 2\zeta(1/2)u$.

4. Classical limit.

In the limit $\theta \to 0$ the Lax hierarchy (5.15) is replaced by the classical equations

$$\partial_{s,j}L^{(-1)} = \{L^{(-1)}, M_{s,j}^{(0)}\}\,,$$

where

$$L^{(-1)} = \frac{s_{mn}}{(m+n\tau)}$$

and $M_{s,j} = M_{s,j}^{(0)} + O(\theta)$. The integrals of motion in this limit survive. We already pointed the form of the Hamiltonian $H = I_{0,2}$ (4.6). The third order integrals take the forms

$$I_{1,3} = \frac{1}{3}\sum_{\sum m_j = \sum n_j = 0}\prod_{j=1}^{3} s_{m_j n_j}\times$$
$$\left(\frac{1}{(m_3+n_3\tau)^2}\sum_j\frac{1}{m_j+n_j\tau} - \frac{1}{2(m_3+n_3\tau)^3}\right),$$

$$I_{2,3} = -\frac{1}{6}\sum_{\sum m_j = \sum n_j = 0}\prod_{j=1}^{3} s_{m_j n_j}\times$$
$$\times\sum_j\frac{1}{m_j+n_j\tau}\,.$$

It turns out that the classical limit in some sense is the same as the rational limit of the basic spectral curve E_τ. Let ω_1, ω_2 be two half-periods of E_τ, $\tau = \omega_2/\omega_1$. The rational limit means that $\omega_1,\ \omega_2 \to \infty$. In this case the Weierstrass function is degenerate as

$$\wp(u) \to \frac{1}{u^2}\,.$$

Consider the double limit $\theta \to 0$, $\omega_1,\ \omega_2 \to \infty$ such that

$$\lim \omega_1\theta = 1, \quad \lim \omega_2\theta = \tau$$

It follows from the definition of the Weierstrass function $\wp(u;\omega_1,\omega_2)$ that in this limit

$$\wp\left[\begin{array}{c} m \\ n \end{array}\right] \to \frac{1}{(m+n\tau)^2}\,.$$

Rescale now the Hamiltonian (4.3) by the dropping the θ^2 multiplier. The new Hamiltonian in the double limit takes the form (4.6).

6. Conclusion

There are three related subjects that are not covered here. The first will be elucidated in separated publications.

• The elliptic rotator on $\mathrm{GL}(N,\mathbb{C})$ can be transformed to the elliptic Calogero-Moser system (ECM) by the symplectic Hecke correspondence [12]. For general orbits we obtain ECM system with spin. The spin degrees of freedom disappear for the most degenerated orbits. The similar approach can be developed for the elliptic rotator on $\mathcal{A}_\theta$. In other words, there should exists a symplectic Hecke correspondence between a special thermodynamical limit of ECM and the elliptic rotator on $\mathcal{A}_\theta$. The general thermodynamical limit of the ECM system leads to ECM system related to $\mathrm{GL}(\infty)$, but here we consider the NCT subgroup and it means that the limit is special. For $\theta = p/N$ after the factorization over the ideal (2.14) we come back to the ECM system for $\mathrm{GL}(N,\mathbb{C})$. I plan to describe ECM system for $\mathcal{A}_\theta$ and its symplectic (Hecke) transformation to the elliptic EAT for $\mathcal{A}_\theta$.

• The EAT can be considered on the central extended algebra $\widehat{Sin}_\theta$ (2.13). For the hydrodynamics ($\mathfrak{J} = \Delta$) this case was investigated in Ref. [15]. Incorporating of the central charge drastically changes the dynamics of EAT. In particular, the integrals (5.10) are no longer the conserved quantities, as well as the Hamiltonian (4.3). For ECM system related to $\mathrm{GL}(N,\mathbb{C})$ this construction leads to the two-dimensional ECM integrable field theory [12,19]. In the group theoretical terms it means that we pass from $\mathrm{GL}(N,\mathbb{C})$ to the central extended loop group $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$.

The same symplectic transformation that maps ECM system to the elliptic EAT is working in the two-dimensional case. In particular, the two-dimensional ECM system for $N = 2$ is mapped to the Landau-Lifshitz equation [12]. In the limit $N \to \infty$ we obtain the symplectic map from the thermodynamical limit of the ECM field theory to the central extended elliptic rotator. The both cases can be described as the systems related to $\widehat{Sin}_\theta$. When $\theta = p/N$ after the factorizing over the ideal (2.15) we come back to the systems considered in [12,19].

• Two different tori are incorporated in our construction - NCT $\mathcal{A}_\theta$ and the basic spectral curve E_τ. In the classical limit they become dual. It seems natural to replace E_τ on another NCT $\mathcal{A}_{\theta'}$. In a general setting it means a generalization of the Hitchin systems to the non-commutative case. One attempt in this direction was done in Ref.[20]. As we have seen, the theta-functions with characteristics are the building blocks for the Lax operators. The analogs of theta-functions on NCT are known [21–23]. We can hope that they will play the essential role in the construction of the Hitchin type systems in the non-commutative situation.

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